THE INNER CORONA ALGEBRA OF A $C_0(X)$-ALGEBRA

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Abstract. Let $A = C(X) \otimes K(H)$, where X is a compact Hausdorff space and $K(H)$ is the algebra of compact operators on a separable, infinite-dimensional Hilbert space. Let A^s be the algebra of strong*-continuous functions from X to $K(H)$. Then A^s/A is the *inner corona algebra of* A. We show that if X has no isolated points then A^s/A is an essential ideal of the corona algebra of A, and $\mathrm{Prim}(A^s/A)$, the primitive ideal space of A^s/A, is not weakly Lindelof. If X is also first countable then there is a natural injection from the power set of X to the lattice of closed ideals of A^s/A. If $X = \beta\mathbf{N} \setminus \mathbf{N}$ and (CH) is assumed then the corona algebra of A is a proper subalgebra of the multiplier algebra of A^s/A. Several of the results are obtained in the more general setting of $C_0(X)$-algebras.

2010 Mathematics Subject Classification: 46L05, 46L08, 46L45 (primary); 46E25, 46J10, 54A35, 54C35, 54D15, 54D35, 54G10 (secondary).

1. Introduction

Let $A = C_0(X)\otimes K(H)$, where $C_0(X)$ is the C*-algebra of complex-valued functions vanishing at infinity on a locally compact Hausdorff space X and $K(H)$ is the algebra of compact operators on a separable infinite-dimensional Hilbert space H. Then it is well known that $M(A)$, the multiplier algebra of A, is isomorphic to the algebra of bounded strong*-continuous functions from X to $B(H)$, the algebra of bounded operators on H [1]. The algebra $M(A)$ contains some obvious ideals arising from the ideal structure of $C_0(X)$ and $B(H)$, but it was shown by Kucerovsky and Ng [13] that if X is an infinite compact metric space then $M(A)$ contains 'non-regular' maximal ideals, a phenomenon that was further investigated in [3].

In studying the ideal structure of $M(A)$, and of the corona algebra $C(A) = M(A)/A$, two natural ideals of $M(A)$ to consider are A^s, the algebra of bounded strong*-continuous functions from X to $K(H)$, and A^b, the algebra of bounded norm-continuous functions from X to $K(H)$. The ideal structure of A^b/A was determined in [17] for the case when X is a finite-dimensional metric space. As far as we know, however, no study has been made of ideals in A^s/A (other than the work on A^b/A and [3, Theorem 5.7]). In this paper we commence such a study. We shall see that A^s/A has a peculiar ideal structure, and is, for instance, very far from being σ-unital. We start by working in the context of a general $C_0(X)$-algebra (see the definition below) but soon find that we have to impose restrictions. The motivating example is always algebras of the form $C_0(X) \otimes K(H)$.

In Section 2 we gather material on $C_0(X)$-algebras, and in Section 3 give the formal definitions of the ideals A^s and A^b of $M(A)$ when A is a general $C_0(X)$-algebra. We seek to identify when these ideals are proper, distinct from each other, and not equal to A. In Section 4 we examine whether A^s/A is an essential ideal in $M(A)/A$ (equivalently whether $\mathrm{Prim}(A^s/A)$, the primitive ideal space of A^s/A, is a dense subset of $\mathrm{Prim}(M(A)/A)$), showing that when $A = C_0(X) \otimes K(H)$ then this is the case if and only if X has no isolated points.

In Section 5 we study A^s/A as a $C_0(X)$-algebra, identifying the range of the base map and showing that if $A = C(\beta\mathbf{N} \setminus \mathbf{N}) \otimes K(H)$, and (CH) is assumed, then there are central multipliers on A^s/A which do not lift to multipliers on A^s (Theorem 5.5). We also show that if $A = C_0(X) \otimes K(H)$ where X is σ-compact, first countable and non-discrete then there is a natural injective map from $2^{X \setminus W}$ (where W is the set of isolated points of X) into the lattice of closed ideals of A^s/A (Theorem 5.6).

In Section 6, we study topological properties of $\mathrm{Prim}(A^s/A)$ showing that if X is σ-compact, infinite and first countable then $\mathrm{Prim}(A^s/A)$ does not satisfy the countable chain condition (Theorem 6.1). Finally, by analyzing the supporting sets of elements of A^s/A, which we show to be meagre, we prove that if $A = C_0(X) \otimes K(H)$ with X σ-compact and without isolated points then $\mathrm{Prim}(A^s/A)$ is not weakly Lindelof (Theorem 6.7).

We are grateful to the referee for a number of helpful comments.

2. Preliminaries

In this section we collect the information that we need on $C_0(X)$-algebras. Recall that a C*-algebra A is a $C_0(X)$-algebra if there is a continuous map ϕ, called the base map, from $\mathrm{Prim}(A)$, the primitive ideal space of A with the hull-kernel topology, to the locally compact Hausdorff space X [20, Proposition C.5]. We will use X_ϕ to denote the image of ϕ in X. Then X_ϕ is completely regular; and if A is σ-unital, X_ϕ is σ-compact and hence normal [3, Section 1].

For $x \in X_\phi$, set

$$J_x := \bigcap \{P \in \mathrm{Prim}(A) : \phi(P) = x\},$$

and for $x \in X \setminus X_\phi$, set $J_x = A$. For $a \in A$, the function $x \to \|a + J_x\|$ $(x \in X)$ is upper semi-continuous [20, Proposition C.10]. The $C_0(X)$-algebra A is said to be *continuous* if, for all $a \in A$, the norm function $x \to \|a + J_x\|$ $(x \in X)$ is continuous. By Lee's theorem [20, Proposition C.10 and Theorem C.26], this happens if and only if the base map ϕ is open. Note that if ϕ is open then X_ϕ is locally compact and is an open subset of βX_ϕ.

Let J be a proper, closed, two-sided ideal of a C^*-algebra A. The quotient map $q_J : A \to A/J$ has a canonical extension $\tilde{q_J} : M(A) \to M(A/J)$ such that $\tilde{q_J}(b)q_J(a) = q_J(ba)$ and $q_J(a)\tilde{q_J}(b) = q_J(ab)$ $(a \in A, b \in M(A))$. We define a proper, closed, two-sided ideal $\tilde{J}$ of $M(A)$ by

$$\tilde{J} = \ker \tilde{q_J} = \{b \in M(A) : ba, ab \in J \text{ for all } a \in A\}.$$

The following proposition was proved in [2, Proposition 1.1].

Proposition 2.1. *Let J be a proper, closed, two-sided ideal of a C^*-algebra A. Then*

(i) *$\tilde{J}$ is the strict closure of J in $M(A)$;*

(ii) *$\tilde{J} \cap A = J$;*

(iii) *if $P \in \mathrm{Prim}(A)$ then $\tilde{P}$ is primitive (and hence is the unique ideal in* $\mathrm{Prim}(M(A))$ *whose intersection with A is P);*

(iv) *$\tilde{J} = \bigcap \{\tilde{P} : P \in \mathrm{Prim}(A)$ and $P \supseteq J\}$ and for all $b \in M(A)$*

$$\|b + \tilde{J}\| = \sup\{\|b + \tilde{P}\| : P \in \mathrm{Prim}(A) \text{ and } P \supseteq J\};$$

(v) *$(A + \tilde{J})/\tilde{J}$ is an essential ideal in $M(A)/\tilde{J}$.*

Furthermore, the map $P \mapsto \tilde{P}$ ($P \in \mathrm{Prim}(A)$) maps $\mathrm{Prim}(A)$ homeomorphically onto a dense, open subset of $\mathrm{Prim}(M(A))$ [15, 4.1.10]. In view of Proposition 2.1(ii), $(A+\tilde{J})/\tilde{J}$ is canonically isomorphic to A/J. Furthermore, if A/J is unital then $(A+\tilde{J})/\tilde{J}$ is a unital essential ideal of $M(A)/\tilde{J}$ and therefore equal to $M(A)/\tilde{J}$, and hence $A+\tilde{J} = M(A)$. (Alternatively, if a is an identity for A modulo J then, for each $b \in M(A)$, one may check that $b - aba \in \tilde{J}$.)

The following proposition, also, was proved in [2, Proposition 1.2].

Proposition 2.2. *Let A be a $C_0(X)$-algebra with base map ϕ. Then ϕ has a unique extension to a continuous map $\overline{\phi} : \mathrm{Prim}(M(A)) \to \beta X$ such that $\overline{\phi}(\tilde{P}) = \phi(P)$ for all $P \in \mathrm{Prim}(A)$. Hence $M(A)$ is a $C(\beta X)$-algebra with base map $\overline{\phi}$ and $\mathrm{Im}(\overline{\phi}) = \mathrm{cl}_{\beta X}(X_\phi)$.*

Now let A be a $C_0(X)$-algebra with base map ϕ and let $\overline{\phi} : \mathrm{Prim}(M(A)) \to \beta X$ be as in Proposition 2.2. For $x \in \mathrm{Im}(\overline{\phi})$, set

$$H_x := \bigcap \{Q \in \mathrm{Prim}(M(A)) : \overline{\phi}(Q) = x\},$$

and for $x \in \beta X \setminus \mathrm{Im}(\overline{\phi})$, set $H_x = M(A)$. Then H_x is a closed two-sided ideal of $M(A)$, and H_x is defined in relation to $(M(A), \beta X, \overline{\phi})$ in the same way that J_x (for $x \in X$) is defined in relation to (A, X, ϕ). It follows that for each $b \in M(A)$, the function $x \to \|b + H_x\|$ ($x \in \beta X$) is upper semi-continuous.

The next proposition was proved in [3, Proposition 2.3].

Proposition 2.3. *Let A be a $C_0(X)$-algebra with base map ϕ, and set $X_\phi = \mathrm{Im}(\phi)$.*
(i) *For all $x \in X$, $J_x \subseteq H_x \subseteq \tilde{J}_x$ and $J_x = H_x \cap A$.*
(ii) *For all $x \in X$, H_x is strictly closed if and only if $H_x = \tilde{J}_x$.*
(iii) *For all $b \in M(A)$, $\|b\| = \sup\{\|b + \tilde{J}_x\| : x \in X_\phi\} = \sup\{\|b + H_x\| : x \in X_\phi\}$.*

If $A = C_0(X) \otimes K(H)$ for a locally compact Hausdorff space X, then we shall assume (unless stated otherwise) that $\phi : \mathrm{Prim}(A) \to X$ is the canonical homeomorphism such that

$$\phi^{-1}(x) = \{f \in C_0(X) : f(x) = 0\} \otimes K(H) \qquad (x \in X).$$

Then it follows from the definition of $\tilde{J}$ before Proposition 2.1 that

$$\tilde{J}_x = \{f \in M(A) : f(x) = 0\}.$$

On the other hand [2, Lemma 1.5(ii)] implies that

$$H_x = \{f \in M(A) : \|f(y)\| \to 0 \text{ as } y \to x\}.$$

For $x \in X_\phi$, the ideals J_x, $\tilde{J}_x$ and H_x depend only on ϕ and X_ϕ and are independent of the ambient space X (see [3, Lemma 2.4] and the discussion which precedes it). We shall therefore often implicitly assume that X is the compact Hausdorff space βX_ϕ, in which case $\overline{\phi}$ maps $\mathrm{Prim}(M(A))$ onto βX_ϕ.

3. The ideals A^s and A^b

Let A be a $C_0(X)$-algebra with base map ϕ. Define

$$A^s := \bigcap_{x \in X_\phi} (A + \tilde{J}_x).$$

Then A^s is a closed two-sided ideal in $M(A)$ and $A^s \supseteq A$. Clearly A^s depends on the particular way in which A is represented as a $C_0(X)$-algebra (there may be many continuous maps from $\mathrm{Prim}(A)$ to X in general). If $A = C_0(X) \otimes K(H)$ for a locally compact Hausdorff space X, then since $\tilde{J}_x = \{f \in M(A) : f(x) = 0\}$, we have that

$$A + \tilde{J}_x = \{f \in M(A) : f(x) \in K(H)\} \qquad (x \in X).$$

Hence A^s is precisely the algebra of bounded strong*-continuous functions from X to $K(H)$ referred to in the introduction.

We begin by characterizing when A^s is a proper ideal in $M(A)$.

Proposition 3.1. *Let A be a $C_0(X)$-algebra with base map ϕ. Then $A^s = M(A)$ if and only if A/J_x is unital for all $x \in X_\phi$.*

Proof. Suppose first that $A^s = M(A)$. Let $x \in X_\phi$. Then $1 \in M(A) = A^s \subseteq A + \tilde{J}_x$. Hence

$$(A + \tilde{J}_x)/\tilde{J}_x \cong A/(A \cap \tilde{J}_x) = A/J_x$$

is unital.

Conversely, suppose that A/J_x is unital for all $x \in X_\phi$. Then $M(A) = A + \tilde{J}_x$ for all $x \in X_\phi$ (see the remark following Proposition 2.1) and hence $M(A) = A^s$. □

For example, let A be the C*-algebra of continuous functions f from the interval $[0,1]$ to the 2×2 complex matrices such that $f(1) = \mathrm{diag}(\lambda(f), 0)$. Then A is non-unital but A is a $C[0,1]$-algebra in a canonical way and A/J_x is unital for all $x \in [0,1]$. Hence $A^s = M(A)$.

Lemma 3.2. *Let A be a $C_0(X)$-algebra with base map ϕ and let $b \in M(A)$. Then $b \in A$ if and only if*

(i) *for all $\epsilon > 0$ the set $\{x \in X_\phi : \|b + H_x\| \geq \epsilon\}$ is compact, and*

(ii) *for all $x \in X_\phi$ there exists $a \in A$ such that $b - a \in H_x$.*

Proof. First, let $b \in A$. Then (ii) holds trivially with $a = b$. Let $\epsilon > 0$ be given. Recall $(A + H_x)/H_x$ is isomorphic to $A/(A \cap H_x) = A/J_x$. Thus

$$\{x \in X_\phi : \|b + H_x\| \geq \epsilon\} = \{x \in X_\phi : \|b + J_x\| \geq \epsilon\} = \phi(\{P \in \mathrm{Prim}(A) : \|b + P\| \geq \epsilon\}).$$

Since ϕ is continuous and the set $\{P \in \mathrm{Prim}(A) : \|b + P\| \geq \epsilon\}$ is compact (see [8, 3.3.7]), it follows that (i) holds.

Conversely, let $b \in M(A)$ and suppose that (i) and (ii) hold. Let $\epsilon > 0$ be given. It is enough to find $c \in A$ such that $\|b - c\| \leq \epsilon$. Set $Y = \{x \in X_\phi : \|b + H_x\| \geq \epsilon\}$. Then Y is compact by (i). Let $y \in Y$. Then by (ii) there exists $a_y \in A$ such that $b - a_y \in H_y$. By the upper semi-continuity of norm functions on βX, there is an open neighbourhood N_y of y in βX such that $\|(b - a_y) + H_x\| < \epsilon$ for all $x \in N_y$. Since Y is compact we may find a finite number of points $y_1, \ldots, y_n \in Y$ such that $Y \subseteq N_{y_1} \cup \ldots \cup N_{y_n}$. Let $h_i : \beta X \to [0,1]$ $(1 \leq i \leq n)$ be continuous functions, with each h_i vanishing off N_{y_i}, such that $\sum_i h_i(x) = 1$

for $x \in Y$ and $\sum_i h_i(x) \leq 1$ for $x \in \beta X \setminus Y$. Set $c = \sum_i \overline{\mu}(h_i) a_{y_i} \in A$ (where the map $\overline{\mu} : C(\beta X) \to ZM(A)$, the centre of $M(A)$, is as described in [2, Proposition 1.2]).

Then for $x \in Y$, $\|(b-c) + H_x\| = \| \sum_i h_i(x)(b - a_{y_i}) + H_x\| < \epsilon$. For $x \in X \setminus Y$, on the other hand,

$$\begin{aligned} \|(b-c) + H_x\| &\leq \|(1 - \sum_i h_i(x))b + H_x\| + \|\sum_i h_i(x)(b - a_{y_i}) + H_x\| \\ &< (1 - \sum_i h_i(x))\epsilon + \sum_i h_i(x)\epsilon = \epsilon. \end{aligned}$$

By Proposition 2.3(iii), $||b-c|| \leq \epsilon$. Hence $b \in A$ as required. □

For a $C_0(X)$-algebra A, we now define

$$A^b := \bigcap_{x \in X_\phi} (A + H_x)$$

(cf. condition (ii) in Lemma 3.2). Then A^b is a closed two-sided ideal in $M(A)$ and $A \subseteq A^b \subseteq A^s$.

If $A = C_0(X) \otimes K(H)$ then $H_x = \{f \in M(A) : \|f(y)\| \to 0 \text{ as } y \to x\}$, as we have observed in Section 2. Hence

$$A + H_x = \{f \in M(A) : \exists T \in K(H) \text{ such that } \|f(y) - T\| \to 0 \text{ as } y \to x\}.$$

From this it follows that A^b is the algebra of bounded norm-continuous functions from X to $K(H)$ (which was the definition of A^b given in the introduction).

We now show that if A is σ-unital then A^b is strictly larger than A unless X_ϕ is compact.

Theorem 3.3. *Let A be a σ-unital $C_0(X)$-algebra with base map ϕ. Then the following are equivalent.*

(i) *$A^b = A$;*

(ii) *X_ϕ is compact.*

Proof. Suppose first that X_ϕ is compact. For $b \in M(A)$ and $\epsilon > 0$ the set $\{x \in X_\phi : \|b + H_x\| \geq \epsilon\}$ is always closed (by upper semi-continuity of norm functions), so it follows from Lemma 3.2 that if $b \in A^b$ then $b \in A$. Hence $A^b = A$.

Conversely, suppose that X_ϕ is non-compact. Then $\beta X_\phi \setminus X_\phi$ is non-empty and since X_ϕ is σ-compact we may write X_ϕ as a strictly increasing union of non-empty compact sets W_n ($n \geq 1$). Let $y \in \beta X_\phi \setminus X_\phi$ and let V_1 be a compact neigbourhood of W_1 in βX_ϕ such that $y \notin V_1$. Then inductively construct a sequence (V_n) of subsets of βX_ϕ such that V_{n+1} is a compact neighbourhood of the compact set $V_n \cup W_{n+1}$ and $y \notin V_{n+1}$. For each $n \geq 1$, let $f_n \in C(\beta X_\phi)$ with $0 \leq f_n \leq 1$, $f_n(y) = 1$, and f_n vanishing on V_n. Set $g_n = f_n|_{X_\phi}$ and $g = \sum_{n=1}^{\infty} g_n$. Then each $x \in X_\phi$ belongs to all but finitely many of the sets W_n, so $g(x)$ is well-defined. Furthermore, if $x \in W_n$ then g_i vanishes in the neighbourhood $V_n \cap X_\phi$ of x for all $i \geq n$, so g is continuous at x. On the other hand, since y lies in the closure of X_ϕ in βX_ϕ, for any $n \geq 1$ we may find $y_n \in X_\phi$ such that $1/2 \leq g_i(y_n)$ for $1 \leq i \leq n$, so that $g(y_n) \geq n/2$. Set $Y = \{y_n\}_{n \geq 1}$ and let $z \in X_\phi$. By the continuity of g, z has a neighbourhood in X_ϕ containing only finitely many points of Y. Hence z has a neighbourhood in X_ϕ disjoint from $Y \setminus \{z\}$. It follows that Y is closed in X_ϕ and that each singleton $\{y_n\}$ is clopen in Y. Thus Y is a countably infinite, relatively discrete, closed subset of X_ϕ.

Let $u \in A$ be strictly positive with $\|u\| \le 1$ and define a continuous function h on Y by $h(y_n) = \|u + J_{y_n}\| > 0$. Since X_ϕ is normal, h has a continuous extension $\overline{h}$ to a bounded, non-negative function on X_ϕ. If $Z(\overline{h})$ is empty, set $f = \overline{h} \wedge 1$. If $Z(\overline{h})$ is non-empty then, since $Z(\overline{h})$ is closed and disjoint from Y, the normality of X_ϕ implies that there exists a continuous function $k : X_\phi \to [0,1]$ such that $k(Z(\overline{h})) = 1$ and $k(Y) = 0$. Set $f = (\overline{h} + k) \wedge 1$. Then either way $f(y_n) = h(y_n)$ for $n \ge 1$ and $Z(f)$ is empty. Let $b \in M(A)$ be the element constructed from f and u by the method of [3, Theorem 2.5]. Then $b \in A^b$ by property (ii) of [3, Theorem 2.5] and the fact that $Z(f) = \emptyset$. On the other hand, since

$$f(y_n) = ||u + J_{y_n}|| = ||u + \tilde{J}_{y_n}||,$$

it follows from property (i) of [3; Theorem 2.5] and the spectral mapping theorem that $||b + \tilde{J}_{y_n}|| = 1$ $(n \ge 1)$. Hence the set $\{x \in X_\phi : \|b + H_x\| \ge 1\}$ contains Y and is therefore non-compact. Thus $b \notin A$ by Lemma 3.2. Hence $A^b \ne A$. □

In studying A^s/A, we are generally working with elements of A^s, so we need to know what happens to such elements in A^s/A. To this end, the following definition is useful.

Definition. Let A be a $C_0(X)$-algebra with base map ϕ. For $b \in A^s$ and $x \in X_\phi$ we say that x is a *point of continuity* for b if $b \in A + H_x$ or, in other words, if there exists $a \in A$ such that $b - a \in H_x$. Let $C(b)$ denote the set of points of continuity of b.

It is immediate that $C(b) = C(b^*)$ and that $C(b) = C(b-a)$ for all $a \in A$; and that for $c \in A^s$, $C(bc) \supseteq C(b)$ and $C(cb) \supseteq C(b)$. It follows from the definitions that $A^b = \{b \in A^s : C(b) = X_\phi\}$, while Lemma 3.2 implies that $b \in A$ if and only if $C(b) = X_\phi$ and for all $\epsilon > 0$ the set $\{x \in X_\phi : \|b + H_x\| \ge \epsilon\}$ is compact. If A is σ-unital and $x \in X_\phi$ is a P-point in X_ϕ then $H_x = \tilde{J}_x$, see the proof of [3, Theorem 4.5], so $x \in C(b)$ for all $b \in A^s$. We shall find the following simple lemma useful.

Lemma 3.4. *Let A be a $C_0(X)$-algebra with base map ϕ. Let $b \in A^s$ and $x \in X_\phi$ with $x \in C(b)$. Suppose that $a \in A$ with $b - a \in \tilde{J}_x$. Then $b - a \in H_x$.*

Proof. Let $c \in A$ with $b - c \in H_x$. Then $(b-c) - (b-a) = a - c \in A \cap \tilde{J}_x = J_x$. Hence $b - a = (b-c) - (a-c) \in H_x + H_x \subseteq H_x$. □

We now introduce a means of constructing elements that will be of considerable importance to us in the rest of the paper. It is taken from [3, Lemma 5.6].

Lemma 3.5. *Let A be a σ-unital, continuous $C_0(X)$-algebra with base map ϕ and suppose that A/J_x is non-unital for all $x \in X_\phi$. Then for each zero set Z in X_ϕ there exists a positive element $c^Z \in A^s$ such that*

(i) $\|c^Z + \tilde{J}_x\| = 0$ *for $x \in Z$;*

(ii) $\|c^Z + \tilde{J}_x\| = 1$ *for $x \in X_\phi \setminus Z$;*

(iii) *for all $x \in X_\phi \setminus Z$ there is a neighbourhood V of x in X_ϕ and an element $a \in A$ such that $c^Z - a \in H_y$ for all $y \in V$.*

It was not stated in [3, Lemma 5.6] that c^Z could be chosen positive, but the proof shows that this is the case.

Lemma 3.6. *Let A be a σ-unital, continuous $C_0(X)$-algebra with base map ϕ and suppose that A/J_x is non-unital for all $x \in X_\phi$. Let Z be a zero set in X_ϕ with boundary ∂Z and let $c^Z \in A^s$ be an element satisfying the properties of Lemma 3.5. Then $C(c^Z) = X_\phi \setminus \partial Z$. Furthermore, for $x \in \partial Z$, $\|c^Z + A + H_x\| \geq 1/2$.*

Proof. For x in the interior of Z it follows from [3, Lemma 4.2] that $c^Z \in H_x$, so $x \in C(c^Z)$. For $x \in X_\phi \setminus Z$ it follows from Lemma 3.5(iii) that $x \in C(c^Z)$. Now let $x \in \partial Z$. Then since $\|c^Z + H_y\| \geq \|c^Z + \tilde{J}_y\| = 1$ for $y \in X_\phi \setminus Z$ it follows that $\|c^Z + H_x\| \geq 1$ by the upper semicontinuity of norm functions on X_ϕ. But $\|c^Z + \tilde{J}_x\| = 0$, so $c^Z \in \tilde{J}_x \setminus H_x$, and hence $x \notin C(c^Z)$ by Lemma 3.4 (with $a = 0$). Thus $C(c^Z) = X_\phi \setminus \partial Z$.

Let $x \in \partial Z$. To estimate $\|c^Z + A + H_x\|$, let $h \in H_x$ and $a \in A$ and suppose for a contradiction that $\|c^Z - a - h\| = \alpha < 1/2$. Then since $c^Z, h \in \tilde{J}_x$ we have that $\|a + \tilde{J}_x\| \leq \alpha$. On the other hand, since $h \in H_x$ there is a neighbourhood V of x in X_ϕ such that $\|h + H_y\| < 1/4 - \alpha/2$ for all $y \in V$. Let $y \in V \setminus Z$. Then $\|c^Z + \tilde{J}_y\| = 1$ by Lemma 3.5(ii), so $\|a + h + \tilde{J}_y\| \geq 1 - \alpha$. Hence

$$\|a + \tilde{J}_y\| > 1 - \alpha - (1/4 - \alpha/2) > 1/2.$$

Choosing a net (y_α) in $V \setminus Z$ with $y_\alpha \to x$, we get a contradiction to the fact that

$$\|a + \tilde{J}_{y_\alpha}\| \to \|a + \tilde{J}_x\| \leq \alpha < 1/2.$$

$\square$

Recall that a $C_0(X)$-algebra A satisfies *spectral synthesis* if $H_x = \tilde{J}_x$ for all $x \in X_\phi$ [4]. Note that if A satisfies spectral synthesis then $A^s = A^b$. To see this, let $b \in A^s$; then for all $x \in X_\phi$, $b \in A + \tilde{J}_x = A + H_x$, so $b \in A^b$. The σ-unital $C_0(X)$-algebras satisfying spectral synthesis were characterized in [4]. Whether spectral synthesis is a necessary condition for $A^s = A^b$ we are not sure, but it is in the case of most interest to us, as we now show.

Theorem 3.7. *Let A be a σ-unital, continuous $C_0(X)$-algebra with base map ϕ and suppose that A/J_x is non-unital for all $x \in X_\phi$. Then the following are equivalent.*
(i) $A^s = A^b$;
(ii) X_ϕ *is discrete;*
(iii) A *satisfies spectral synthesis.*

Proof. The equivalence of (ii) and (iii) was established in [3, Theorem 4.5], while (iii)⇒(i) has already been observed.

Now suppose that (iii) fails, so that there exists $y \in X_\phi$ such that $H_y \neq \tilde{J}_y$. Then y is not a P-point in X_ϕ by [3, Theorem 4.5], so there is a zero set Z in X_ϕ such that y lies in the boundary of Z. Let $c^Z \in A^s$ be an element as in Lemma 3.5. Then $y \notin C(c^Z)$ by Lemma 3.6 and hence $c^Z \notin A^b$. Thus (i)⇒(iii). $\square$

Combining Theorem 3.3 and Theorem 3.7, we see that under the hypotheses of Theorem 3.7, $A = A^s$ if and only if X_ϕ is finite. Thus the inner corona algebra A^s/A is a non-zero ideal in the corona algebra $M(A)/A$ unless X_ϕ is finite. In the next section we shall consider whether A^s/A is an essential ideal in $M(A)/A$.

4. When is A^s/A an essential ideal in $M(A)/A$?

In this section we consider the density of $\mathrm{Prim}(A^s/A)$ in $\mathrm{Prim}(C(A))$, where $C(A)$ is the corona algebra of A. Suppose that A is a $C_0(X)$-algebra with base map ϕ and that x is an isolated point of X_ϕ with A/J_x non-unital. Let V be the inverse image of x under $\overline{\phi}$. Then V is a clopen subset of $\mathrm{Prim}(M(A))$ and V has non-empty intersection with $\mathrm{Prim}(C(A))$ because $A+\tilde{J}_x$ is a proper ideal of $M(A)$ containing H_x. This intersection is relatively open in $\mathrm{Prim}(C(A))$ and is disjoint from $\mathrm{Prim}(A^s/A)$ since $A^s \subseteq A+\tilde{J}_x = A+H_x$. Thus the closure of $\mathrm{Prim}(A^s/A)$ in $\mathrm{Prim}(C(A))$ is contained in the inverse image of $\beta X_\phi \setminus Y$ under $\overline{\phi}$, where Y is the set of isolated points of X_ϕ. In particular if all the quotients A/J_x $(x \in X_\phi)$ are non-unital then a necessary condition for A^s/A to be an essential ideal in $C(A)$ is that X_ϕ should have no isolated points.

On the other hand, if $A = C_0(X)\otimes K(H)$ where X is a locally compact Hausdorff space without isolated points then A^s/A is an essential ideal in $C(A)$, as we now show.

Theorem 4.1. *Let $A = C_0(X)\otimes K(H)$ where X is a locally compact Hausdorff space. Then A^s/A is an essential ideal in $C(A)$ if and only if X is without isolated points.*

Proof. We have seen the necessity of the condition that X be without isolated points. For the sufficiency of this condition, let $b \in M(A)\setminus A^s$ with $b \geq 0$. It is enough to find $c \in A^s$ such that $bc \notin A$. Since $b \notin A^s$, there exists $x \in X$ with $b(x) \in B(H)\setminus K(H)$. Without loss of generality we may assume that $\|b(x)+K(H)\| = 1$. Let E be the spectral projection of $b(x)$ corresponding to $[1/2,\infty)$ and define $H_0 = E(H)$. Then H_0 is an infinite-dimensional reducing subspace of $b(x)$ such that $\|b(x)\xi\| \geq 1/2$ for all $\xi \in H_0$ with $\|\xi\| = 1$. Let $\{\xi_i\}_{i\geq 1}$ be an orthonormal basis for H_0, and for each $i \geq 1$ let p_i be the 1-dimensional projection in $B(H)$ with range ξ_i. Then $\|b(y)\xi_i - b(x)\xi_i\| \to 0$ for each $i \geq 1$ as $y \to x \in X$. Hence for each $i \geq 1$, there is a neighbourhood U_i of x such that $\|b(y)\xi_i\| > 1/4$ for all $y \in U_i$.

Let V be a compact neighbourhood of x, and inductively define points $y_i \in V \cap U_i$ together with neighbourhoods Y_i and V_i of y_i having the following properties: $y_i \in Y_i \subseteq V_i$; Y_i is compact and V_i is open; $V_i \cap V_j = \emptyset$ for $i \neq j$; and x does not lie in the closure of V_i. Since X has no isolated points, there exists $y_1 \in V \cap U_1$ with $y_1 \neq x$, and since X is Hausdorff and locally compact there exists an open neighbourhood V_1 of y_1 such that x is not in the closure of V_1 and there exists a compact neighbourhood Y_1 of y_1 contained in V_1. Now given $y_1,\dots,y_n$ with neighbourhoods Y_i and V_i $(1 \leq i \leq n)$ satisfying the given properties, $V \cap U_{n+1} \setminus (\overline{V_1}\cup\cdots\cup\overline{V_n})$ is a neighbourhood of x and therefore contains an element y_{n+1} not equal to x. Let V_{n+1} be an open neighbourhood of y_{n+1} disjoint from $\overline{V}_1 \cup \ldots \cup \overline{V}_n \cup \{x\}$ and such that x does not lie in the closure of V_{n+1}. Let Y_{n+1} be a compact neighbourhood of y_{n+1} contained in V_{n+1}. Proceeding inductively, we obtain the desired sequence of points and neighbourhoods. Note that $\|b(y_i)\xi_i\| > 1/4$ for all $i \geq 1$.

For each $i \geq 1$, let $f_i : X \to [0,1]$ be a continuous function such that $f_i(y_i) = 1$ and f_i is supported in Y_i, and define $c_i \in A^+$ by $c_i(y) = f_i(y)p_i$ $(y \in X)$. We claim that the function $c : X \to B(H)^+$ given by $c(y) = \sum_{i=1}^\infty c_i(y)$ defines a (positive) element of A^s. To see this, note first that for each $y \in X$, the sum defining $c(y)$ has at most one non-zero term. Thus the function c is defined. To check strong*-continuity, let ζ be any non-zero vector in H. Then the map $y \mapsto \|c(y)\zeta\|$ is clearly continuous at all points $y \in \bigcup_{i=1}^\infty V_i$. Suppose that $y \in X \setminus \bigcup_{i=1}^\infty V_i$ and let $\epsilon > 0$ be given. Then there exists $n \geq 1$ such that

$\sum_{i=n}^{\infty} |\langle \xi_i, \zeta\rangle|^2 < \epsilon^2$. Hence $\|c(z)\zeta\| < \epsilon$ for all z in the open neighbourhood $X \setminus \bigcup_{i=1}^{n} Y_i$ of y. Thus c is strong*-continuous at y, so $c \in A^s$.

On the other hand, note that for $i \geq 1$,

$$\|bc(y_i)\| \geq \|b(y_i)c_i(y_i)\xi_i\| = \|b(y_i)p_i\xi_i\| > 1/4.$$

Since the sequence $(y_i)_{n\geq 1}$ is contained in the compact set V, it has a cluster point $y \in V$. Then $y \notin \bigcup_{i=1}^{\infty} V_i$, so $bc(y) = 0$. Hence bc is not norm-continuous at y, so $bc \notin A$, as required. □

Theorem 4.1 raises the question of whether A^s/A is an essential ideal in $C(A)$ in the case where A is a $C_0(X)$-algebra such that X_ϕ has no isolated points and A/J_x is non-unital for all $x \in X$.

5. A^s/A as a $C_0(X)$-algebra

Let A be a $C_0(X)$-algebra with base map ϕ and recall that ϕ has a canonical extension $\overline{\phi} : \mathrm{Prim}(M(A)) \to \beta X_\phi$. Thus, with the usual identifications, we may consider $\overline{\phi}|_{\mathrm{Prim}(A^s)}$ and $\overline{\phi}|_{\mathrm{Prim}(A^s/A)}$. The latter of these maps we shall denote by ψ. Thus A^s/A is a $C(\beta X_\phi)$-algebra with base map ψ. We begin this section by investigating the properties of ψ.

Set $X_\psi = \mathrm{Im}\,\psi$. Note that for $x \in \beta X_\phi$, $x \in X_\psi$ if and only if there exists $Q \in \mathrm{Prim}(M(A))$ such that $Q \supseteq A$, $Q \not\supseteq A^s$, and $\overline{\phi}(Q) = x$, if and only if there exists $Q \in \mathrm{Prim}(M(A))$ such that $Q \supseteq A + H_x$ and $Q \not\supseteq A^s$, if and only if $A^s \not\subseteq A + H_x$.

For $b \in A^s$, let

$$\mathrm{coz}(b + A) := \{x \in X_\psi : b \notin H_x + A\},$$

the *supporting set* of $b + A \in A^s/A$ (note that we do not take the closure in this definition), and let

$$\mathrm{coz}_\infty(b) := \{x \in \beta X_\phi \setminus X_\phi : b \notin H_x\}.$$

The next lemma relates $\mathrm{coz}(b + A)$ to $\mathrm{coz}_\infty(b)$ and to $C(b)$.

Lemma 5.1. *Let A be a $C_0(X)$-algebra with base map ϕ. Then for $b \in A^s$, $\mathrm{coz}(b + A) = \mathrm{coz}_\infty(b) \cup (X_\phi \setminus C(b))$.*

Proof. For $x \in X_\phi$ it is immediate from the definitions that $x \in C(b)$ if and only $x \notin \mathrm{coz}(b + A)$. For $x \in \beta X_\phi \setminus X_\phi$ it is immediate from the definitions, since $H_x \supseteq A$, that $x \in \mathrm{coz}(b + A)$ if and only if $x \in \mathrm{coz}_\infty(b)$. □

Theorem 5.2. *Let A be a σ-unital, continuous $C_0(X)$-algebra with base map ϕ and suppose that X_ϕ is infinite and that A/J_x is non-unital for all $x \in X_\phi$. Then A^s/A is a non-trivial $C(\beta X_\phi)$-algebra with base map ψ, and $X_\psi = \beta X_\phi \setminus W$ where W is the set of P-points in X_ϕ.*

Proof. Since X_ϕ is infinite, it follows from the remark after Theorem 3.7 that $A^s \neq A$. Now let Z be the empty set and let c^Z be an element as in Lemma 3.5. Then

$$\|c^Z + H_x\| \geq \|c^Z + \tilde{J}_x\| \geq 1 \qquad (x \in X_\phi)$$

and so $\|c^Z + H_x\| \geq 1$ for $x \in \beta X_\phi \setminus X_\phi$ by upper semi-continuity of norm functions. Hence

$$\mathrm{coz}(c^Z + A) \supseteq \mathrm{coz}_\infty(c^Z) = \beta X_\phi \setminus X_\phi$$

and so $X_\psi \supseteq \beta X_\phi \setminus X_\phi$.

Now let $x \in X_\phi$. If x is a P-point in X_ϕ then, as we noted earlier, $x \in C(b)$ for all $b \in A^s$ and hence $A^s \subseteq A + H_x$. Thus $x \notin X_\psi$. On the other hand, if x is a non-P-point in X_ϕ then it follows as in the proof of Theorem 3.7 that there exists $b \in A^s$ with $x \notin C(b)$. Hence $A^s \not\subseteq A + H_x$ so $x \in X_\psi$. Thus $X_\psi = \beta X_\phi \setminus W$. $\square$

In the context of Theorem 5.2, if A is separable then X_ϕ is a locally compact metrizable space and hence every P-point in X_ϕ is an isolated point. Thus the set W of P-points in X_ϕ is open in βX_ϕ, from which it follows that X_ψ is compact by Theorem 5.2. On the other hand, if A is non-separable and X_ϕ has a non-isolated P-point x then every compact neighbourhood of x must contain a non-P-point (recall that a compact P-space is finite [10, 4K]) and hence X_ψ is non-compact. This implies that $\mathrm{Prim}(A^s/A)$ is also non-compact. The topological properties of $\mathrm{Prim}(A^s/A)$ are considered further in the next section.

We now give some illustrative examples. The first three are routine (from our present point of view) but the fourth one is of considerable interest.

Example 5.3. (i) Let $A = C_0(\mathbf{N}) \otimes K(H)$. Then $X_\phi = \mathbf{N}$ and $X_\psi = \beta\mathbf{N} \setminus \mathbf{N}$.
(ii) Let $A = C(\mathbf{N} \cup \{\infty\}) \otimes K(H)$. Then $X_\phi = \mathbf{N} \cup \{\infty\}$ and X_ψ is the singleton $\{\infty\}$.
(iii) Let $A = C(\beta\mathbf{N}) \otimes K(H)$. Then $X_\phi = \beta\mathbf{N}$ and again $X_\psi = \beta\mathbf{N} \setminus \mathbf{N}$.

Example 5.4. Set $\mathbf{N}^* = \beta\mathbf{N} \setminus \mathbf{N}$ and let $A = C(\mathbf{N}^*) \otimes K(H)$. Then $X_\psi = \mathbf{N}^* \setminus W$ where W is the set of P-points in $\mathbf{N}^*$. In 1978, Shelah proved in his famous P-point independence theorem that there are models of set theory in which W is empty (see e.g. [21]), and hence for which X_ψ is the compact Hausdorff space $\mathbf{N}^*$.

On the other hand, if (CH) is assumed then it was shown by Rudin [18] that W is dense in $\mathbf{N}^*$, implying that X_ψ is nowhere locally compact. (Note, however, that X_ψ itself is dense in $\mathbf{N}^*$ in all models of set theory because otherwise W would contain a non-empty open subset of $\mathbf{N}^*$, which in turn would contain a compact P-space with non-empty interior in $\mathbf{N}^*$, and this would have to be finite, contradicting the non-existence of isolated points in $\mathbf{N}^*$). Furthermore, under (CH), if $p \in \mathbf{N}^*$ then $\mathbf{N}^* \setminus \{p\}$ is not C*-embedded in $\mathbf{N}^*$; that is, there exist continuous bounded functions on $\mathbf{N}^* \setminus \{p\}$ which do not extend continuously to $\mathbf{N}^*$ [9]. Fixing a P-point $p \in \mathbf{N}^*$, let f be such a function on $\mathbf{N}^* \setminus \{p\}$. Then $f \circ \psi$ defines a continuous bounded function on $\mathrm{Prim}(A^s/A)$, and hence, by the Dauns-Hofmann theorem, a central multiplier on A^s/A. We shall show that there is no $b \in M(A)$ such that $b + A$ induces the same central multiplier on A^s/A as $f \circ \psi$.

Recall that for any closed ideal I in a C*-algebra A there is a canonical *-homomorphism $A \to M(I)$ which is injective if and only if I is an essential ideal in A [15, 3.12.8].

Theorem 5.5. *Let $A = C(\mathbf{N}^*) \otimes K(H)$ and assume (CH). Then the corona algebra $M(A)/A$ is (canonically isomorphic to) a proper subalgebra of $M(A^s/A)$.*

Proof. Let p be a P-point in $\mathbf{N}^*$ and let f be as in Example 5.4. First note that for all $b \in M(A)$, the function $x \mapsto b(x)$ $(x \in \mathbf{N}^*)$ is norm-continuous at p. To see this, consider the element $c := b - b(p)1$. Then $c(p) = 0$, so $c \in \tilde{J}_p$. But since p is a P-point, $\tilde{J}_p = H_p$ [3, Theorem 4.5]. Thus the function $x \mapsto \|c(x)\|$ $(x \in \mathbf{N}^*)$ tends to zero as $x \to p$, so $\|b(x) - b(p)\| \to 0$.

Now suppose for a contradiction that there exists $b \in M(A)$ such that $b + A$ induces the same central multiplier on A^s/A as $f \circ \psi$; or in other words that for $c \in A^s$ and $x \in X_\psi = \mathbf{N}^* \setminus W$,

$$bc + (A + H_x) = cb + (A + H_x) = f(x)c + (A + H_x). \qquad (*)$$

For $y \in \mathbf{N}^*$ with $y \neq p$, let $g : \mathbf{N}^* \to [0,1]$ be a continuous function such that $g(p) = 0$ and g takes the constant value 1 in a neighbourhood of y. Then $gf \in C^b(\mathbf{N}^*) \subseteq M(A)$. Let $c \in A^s$ and $x \in X_\psi$. Then $g(b - f(x)1)c \in A + H_x$ by $(*)$. But

$$\|((gf)c - gf(x)c)(z)\| \leq \|g\|_\infty \|c\|_\infty |f(z) - f(x)| \to 0$$

as $z \to x$ and so $(gf)c - gf(x)c \in H_x$. Hence $(gb - gf)c \in A + H_x$. On the other hand, for $x \in W$, $(gb - gf)c \in A + \tilde{J}_x = A + H_x$ since x is a P-point. Hence $(gb - gf)c \in A + H_x$ for all $x \in \mathbf{N}^*$ so $(gb - gf)c \in A$ by Theorem 3.3. Similarly $c(gb - gf) \in A$. But this implies that $gb - gf \in A$ by Theorem 4.1, so in particular there exists $k(y) \in K(H)$ such that $b(y) = gb(y) = f(y)1 + k(y)$. Thus we have shown that, for each $y \in \mathbf{N}^* \setminus \{p\}$, there exists $k(y) \in K(H)$ such that $b(y) = f(y)1 + k(y)$.

Since f does not extend continuously to $\mathbf{N}^*$, and since X_ψ is dense in $\mathbf{N}^*$, there exists nets (y_α) and (y_β) in X_ψ converging to the P-point p such that $f(y_\alpha) \to \gamma$ and $f(y_\beta) \to \delta$ where $\gamma, \delta \in \mathbf{R}$ with $\gamma \neq \delta$. By subtracting an appropriate scalar and re-scaling, we may assume that $\gamma = 0$ and $\delta = 1$. Using the P-point property of p, we have that $f(y_\alpha) = 0$ eventually and $f(y_\beta) = 1$ eventually. Hence, by the first paragraph, we have that $(k(y_\alpha))$ is a norm-convergent net of compact operators with limit $b(p)$, while $(1 + k(y_\beta))$ is a norm-convergent net of operators each of which is distance 1 from the set of compact operators, but also with limit $b(p)$. Thus $b(p)$ is simultaneously a compact operator and distance 1 from the compact operators, which gives a contradiction.

This shows that the multiplier $f \circ \psi$ is not induced by any element of $M(A)$, and thus the canonical homomorphism from $M(A)$ to $M(A^s/A)$ is not surjective. Combining this with Theorem 4.1, it follows that the corona algebra $C(A)$ is (canonically isomorphic to) a proper subalgebra of $M(A^s/A)$. □

Our final result in this section is on the multiplicity of ideals in A^s/A. Recall that $A^s = \bigcap_{x \in X_\phi}(A + \tilde{J}_x)$, that $A^b = \bigcap_{x \in X_\phi}(A + H_x)$ and that W is the set of P-points in X_ϕ. Note that if X_ϕ is first countable then every P-point in X_ϕ is isolated.

Theorem 5.6. *Let A be a σ-unital, continuous $C_0(X)$-algebra with base map ϕ. Suppose that X_ϕ is first countable and non-discrete, and that A/J_x is non-unital for all $x \in X_\phi$. For each subset Y of the non-empty set $X_\phi \setminus W$, let $I(Y) = A^s \cap \bigcap_{x \in Y}(A + H_x)$. Then the map $Y \mapsto I(Y)$ is injective from the power set $2^{X_\phi \setminus W}$ to the lattice of closed ideals of A^s which contain A^b.*

Proof. Since X_ϕ is first countable and non-discrete, $X_\phi \setminus W$ is non-empty. If Y_1 and Y_2 are distinct subsets of $X_\phi \setminus W$ then without loss of generality there exists $y \in Y_1 \setminus Y_2$. Since X_ϕ is first countable, $Z := \{y\}$ is a zero set in X_ϕ. Furthermore, $\partial Z = Z$ since y is not an isolated point of X_ϕ. Let c^Z be an element with the properties of Lemma 3.5. Then by Lemma 3.6, $C(c^Z) = X_\phi \setminus Z$, so $c^Z \in I(Y_2) \setminus I(Y_1)$. Thus $I(Y_1)$ and $I(Y_2)$ are distinct closed ideals, as required. □

From the famous theorem of Arhangel'skii in 1969 [7], it follows that a locally compact, Lindelof, first countable Hausdorff space without isolated points has cardinality $\mathbf{c}$. Thus if

A satisfies the hypotheses of Theorem 5.6, and if X_ϕ has no isolated points, then A^s/A has at least $2^{\mathfrak{c}}$ closed ideals. This is the case, for example, if $A = C[0,1] \otimes K(H)$. It follows that $\mathrm{Prim}(A^s/A)$ must be large enough to distinguish all these ideals. To this subject we now turn.

6. Topological properties of $\mathrm{Prim}(A^s/A)$

In this final section we give further consideration to the topological space $\mathrm{Prim}(A^s/A)$. Suppose that A is a σ-unital continuous $C_0(X)$-algebra with A/J_x non-unital for all $x \in X_\phi$ and X_ϕ infinite (so that A^s/A is non-zero by Theorem 3.3 and Theorem 3.7). Then, provided that X_ϕ is first countable, we show that $\mathrm{Prim}(A^s/A)$ does not satisfy the *countable chain condition*: in other words, $\mathrm{Prim}(A^s/A)$ admits uncountable families of pairwise disjoint non-empty open subsets (Theorem 6.1). In particular $\mathrm{Prim}(A^s/A)$ is non-separable. This gives one indication of how large $\mathrm{Prim}(A^s/A)$ is.

We saw after Theorem 5.2 that if A is a σ-unital continuous $C_0(X)$-algebra with A/J_x non-unital for all $x \in X_\phi$ and if X_ϕ has a non-isolated P-point then $\mathrm{Prim}(A^s/A)$ is not compact. We left open, however, the general question of the compactness of $\mathrm{Prim}(A^s/A)$. Our second main result is to show that if X_ϕ has no isolated points and if either A is a separable, continuous $C_0(X)$-algebra with A/J_x non-unital for all $x \in X_\phi$, or A has the form $C_0(X) \otimes K(H)$ where X is σ-compact, then $\mathrm{Prim}(A^s/A)$ is not weakly Lindelof (Theorem 6.7). To prepare for this, we study the supporting sets of elements of A^s/A, showing that they are meagre in βX_ϕ, and are meagre in X_ψ too if X_ϕ has no isolated points (Theorem 6.5 and Corollary 6.6).

Theorem 6.1. *Let A be a σ-unital continuous $C_0(X)$-algebra with A/J_x non-unital for all $x \in X_\phi$. Suppose that X_ϕ is infinite and first countable. Then* $\mathrm{Prim}(A^s/A)$ *does not satisfy the countable chain condition.*

Proof. Let S be the subset of X_ϕ consisting of the isolated points of X_ϕ. First suppose that S is finite. Then $X_\phi \setminus S$ is a non-empty clopen subset of the locally compact space X_ϕ. Hence there exists a non-empty open subset U of $X_\phi \setminus S$ whose closure V in X_ϕ is a compact subset of $X_\phi \setminus S$. Then V is a non-empty compact Hausdorff space without isolated points, and since the complement of a non-isolated point is a dense open set, it follows immediately by the Baire property that V is uncountable. Since X_ϕ is first countable, each singleton subset of V is a zero set in X_ϕ. For $z \in V$, set $Z = \{z\}$ and let c^Z be an element of A^s with the properties of Lemma 3.5. Let $g_z : X_\phi \to [0,1]$ be a continuous function with compact support such that $g_z(z) = 1$ and let f_z be the unique continuous extension to βX_ϕ. By the Dauns-Hofmann Theorem, the continuous function $f_z \circ \overline{\phi}$ on $\mathrm{Prim}(M(A))$ induces an element k_z in the centre of $M(A)$ such that $k_z + H_x = f_z(x)$ for all $x \in \beta X_\phi$. Set $d_z = k_z c^Z \in A^s$. Since $f_z(z) = 1$, it follows from Lemma 3.6 that $C(d_z) = C(c^Z) = X_\phi \setminus \{z\}$. In particular, $d_z \notin A$. Hence the open set

$$U_z := \{P \in \mathrm{Prim}(A^s/A) : d_z + A \notin P\}$$

is non-empty.

We now show that the sets U_z $(z \in V)$ are pairwise disjoint. Let z_1 and z_2 be distinct elements of V and let $b \in A^s$. Since $C(d_{z_i}) = X_\phi \setminus \{z_i\}$ and $z_1 \neq z_2$, $d_{z_1} b d_{z_2} \in A + H_x$ for all

$x \in X_\phi$. Let $\epsilon > 0$. The set

$$\{x \in X_\phi : \|d_{z_1} b d_{z_2} + H_x\| \geq \epsilon\}$$

is closed in X_ϕ (by upper semi-continuity) and is contained in $\mathrm{supp}(g_1) \cup \mathrm{supp}(g_2)$ and hence is compact. It follows from Lemma 3.2 that $d_{z_1} b d_{z_2} \in A$. Thus $d_{z_1} A^s d_{z_2} \subseteq A$ and hence $U_{z_1} \cap U_{z_2} = \emptyset$, as required. Since the sets U_z $(z \in V)$ form an uncountable family of pairwise disjoint open sets, $\mathrm{Prim}(A^s/A)$ does not satisfy the countable chain condition.

Next suppose that the set S of isolated points of X_ϕ is infinite and has infinite intersection with some compact subset of X_ϕ. Using the first countability of X_ϕ we may find a convergent sequence $(x_i)_{i\geq 1}$ of distinct points of S with limit $x \in X_\phi$. Then $Y := \{x_i : i \geq 1\} \cup \{x\}$ is closed in X_ϕ. It is well known that there exists an uncountable family $\mathcal{F}$ of infinite pairwise almost disjoint subsets of $\mathbf{N}$ (where *almost disjoint* means having finite intersection). Then for each $F' \in \mathcal{F}$ the set $\{x_i : i \in F'\}$ is a countable family of isolated points in X_ϕ and hence is a cozero set in X_ϕ. Let $F := X_\phi \setminus F'$ be the corresponding zero set in X_ϕ and let c^F be an element of A^s with the properties of Lemma 3.5 relative to F. Then $\|c^F + \tilde{J}_{x_i}\| = 1$ for $i \in F'$ and $c^F \in \tilde{J}_y$ for $y \in F$; and by Lemma 3.6, $C(c^F) = X_\phi \setminus \{x\}$, since F' is infinite. In particular, $c^F \notin A$.

Let $F', G' \in \mathcal{F}$ with $F' \neq G'$, and let $b \in A^s$. Then $c^F b c^G \in \tilde{J}_y$ for all $y \in F \cup G$, and $F \cup G$ is cofinite, and hence open, in X_ϕ. Thus $c^F b c^G \in H_y$ for all $y \in F \cup G$ by [2, Lemma 1.5(i)]. Since $C(c^F) = X_\phi \setminus \{x\}$ and $x \in F \cup G$, $c^F b c^G \in A + H_y$ for all $y \in X_\phi$. It follows that $c^F b c^G \in A$ by Lemma 3.2, and thus $c^F A^s c^G \subseteq A$. Hence the elements $c^F + A$ are nonzero in A^s/A and the non-empty open sets

$$\{Q \in \mathrm{Prim}(A^s/A) : c^F + A \notin Q\}$$

are pairwise disjoint, so again $\mathrm{Prim}(A^s/A)$ does not satisfy the countable chain condition.

Finally suppose that the set S is infinite but has finite intersection with each compact subset of X_ϕ. Since X_ϕ is locally compact, S is a clopen subset of X_ϕ, so we have that $\beta X_\phi = \beta S \cup \beta(X_\phi \setminus S)$, the disjoint union of two clopen subsets. Clearly $S \subseteq W$, in the notation of Theorem 5.2, and thus $\beta S \setminus S$ is a clopen subset of X_ψ. But S is countable since X_ϕ is Lindelof, so $\beta S \setminus S$ does not satisfy the countable chain condition (using the uncountable family of almost disjoint subsets of the previous paragraph); and $\beta S \setminus S$ is a clopen subset of the image of $\mathrm{Prim}(A^s/A)$ under the continuous map ψ. Hence $\mathrm{Prim}(A^s/A)$ does not satisfy the countable chain condition. □

With c^F as one of the elements constructed in the third paragraph of the proof above, note that $\mathrm{coz}_\infty(c^F)$ is empty by [3, Lemma 4.2]. Hence $\mathrm{coz}(c^F + A) = \{x\}$ by Lemma 5.1. It follows that the elements $\{c^F + A\}_{F' \in \mathcal{F}}$ form an uncountable family of orthogonal elements of A^s/A with the singleton $\{x\}$ as their common supporting set. Thus $\psi^{-1}(x)$ contains in its interior the entire uncountable family of disjoint open subsets of $\mathrm{Prim}(A^s/A)$ exhibited in that paragraph. We should also note that, since $\mathrm{Prim}(A^s/A)$ is homeomorphic to an open subset of $\mathrm{Prim}(M(A)/A)$, it follows under the hypotheses of Theorem 6.1 that $\mathrm{Prim}(M(A)/A)$ does not satisfy the countable chain condition either.

We now turn to the study of the supporting sets of elements of A^s/A. Part (ii) of the following lemma explains the name 'point of continuity' for $y \in C(b)$.

Lemma 6.2. *Let A be a continuous $C_0(X)$-algebra with base map ϕ.*

(i) *For* $b \in M(A)$*, the map* $x \mapsto \|b + \tilde{J}_x\|$ $(x \in X_\phi)$ *is lower semi-continuous.*

(ii) *Let* $b \in A^s$*. Then* $y \in C(b)$ *if and only if the map* $x \mapsto \|(b-a) + \tilde{J}_x\|$ $(x \in X_\phi)$ *is continuous at* y *for all* $a \in A$*.*

Proof. (i) Let $\epsilon \geq 0$. Then $\{x \in X_\phi : \|b + \tilde{J}_x\| > \epsilon\} = \phi(\{P \in \mathrm{Prim}(A) : \|b + \tilde{P}\| > \epsilon\})$, and this latter set is open, being the image of an open set [8, Proposition 3.3.2] under an open map.

(ii) First suppose that $y \in C(b)$ and let $a \in A$. Since $C(b) = C(b-a)$ we may assume, replacing b by $b-a$, that $a = 0$. Then there exists $c \in A$ with $b - c \in H_y$. Hence

$$\|b + \tilde{J}_y\| = \|((b-c)+c) + \tilde{J}_y\| = \|c + \tilde{J}_y\|.$$

Let $\epsilon > 0$. Then by upper semi-continuity (at H_y) and continuity (since $c \in A$) there is an open subset V of X_ϕ containing y such that $\|(b-c)+H_x\| < \epsilon/2$ and $\|c+J_x\| < \|c+J_y\|+\epsilon/2$ for all $x \in V$. Then for $x \in V$,

$$\|b + \tilde{J}_x\| \leq \|b + H_x\| \leq \|(b-c) + H_x\| + \|c + H_x\|$$

$$< \epsilon/2 + \|c + J_y\| + \epsilon/2 = \|c + \tilde{J}_y\| + \epsilon.$$

Thus the norm function of b is upper semi-continuous at y, and hence continuous by (i).

Conversely, suppose that the map $x \mapsto \|(b-a) + \tilde{J}_x\|$ $(x \in X_\phi)$ is continuous at y for all $a \in A$. Let $a \in A$ such that $b - a \in \tilde{J}_y$. Let $\epsilon > 0$. Then by continuity there is a neighbourhood V of y in X_ϕ such that $\|(b-a) + \tilde{J}_x\| < \epsilon$ for all $x \in V$. It follows from [3, Lemma 4.2] that $\|(b-a) + H_y\| \leq \epsilon$. Hence $b - a \in H_y$, so $y \in C(b)$. □

The next result is a well-known consequence of the method of [8, B18]. For the reader's convenience, we include the proof here.

Proposition 6.3. *Let* X *be a Baire space and* $f : X \to [0,1]$ *be a lower semi-continuous function. Then the set of points of continuity of* f *is a dense* G_δ *in* X*.*

Proof. For $x \in X$ we define the *oscillation* $\omega(x)$ of f at x by $\omega(x) = \inf_U(\alpha(U) - \beta(U))$, where the infimum is taken over open neighbourhoods U of x and where $\alpha(U) = \sup\{f(x) : x \in U\}$ and $\beta(U) = \inf\{f(x) : x \in U\}$. Then f is continuous at x if and only if $\omega(x) = 0$. Let (x_α) be a net in X with limit x and let $\epsilon > 0$. Then there is an open set U containing x such that $\alpha(U) - \beta(U) < \omega(x) + \epsilon$. Hence eventually, $\omega(x_\alpha) < \omega(x) + \epsilon$. Thus the function $x \mapsto \omega(x)$ is upper semi-continuous on X. For $n \in \mathbf{N}$, set $X_n = \{x \in X : \omega(x) \geq 1/n\}$. Then X_n is closed by upper semi-continuity of ω. We show that X_n has empty interior. Suppose to the contrary that the interior U of X_n is non-empty. Let $x \in U$ with $f(x) > \alpha(U) - 1/2n$. Then by the lower semi-continuity of f, x has an open neighbourhood contained in U in which $\alpha(U) - 1/2n < f(y) \leq \alpha(U)$, and hence $\omega(x) < 1/2n$, a contradiction. Thus X_n has empty interior. Finally,

$$\{x \in X : \omega(x) = 0\} = \bigcap_{n \geq 1} X \setminus X_n$$

which is a dense G_δ in X. □

We now show that in certain circumstances $C(b)$ is a dense G_δ in X_ϕ. Recall that if A is a continuous $C_0(X)$-algebra then X_ϕ is a locally compact Hausdorff space.

Lemma 6.4. *Let A be a continuous $C_0(X)$-algebra with base map ϕ and let $b \in A^s$. Suppose that either* (i) *A is separable or* (ii) *$A = C_0(X) \otimes K(H)$ where X is a σ-compact, locally compact Hausdorff space. Then $C(b)$ is a dense G_δ in X_ϕ.*

Proof. Case (i). Let $\{a_i\}_{i\geq 1}$ be a countable dense subset of A. For each $i \geq 1$, let $C'(b-a_i)$ be the set of points of continuity of the function $x \mapsto \|(b-a_i)+\tilde{J}_x\|$ $(x \in X_\phi)$. Then $C'(b-a_i)$ is a dense G_δ in X_ϕ by Lemma 6.2(i) and Proposition 6.3. Thus $C := \bigcap_{i=1}^{\infty} C'(b-a_i)$ is a dense G_δ in X_ϕ since X_ϕ is a locally compact, Hausdorff space, and hence a Baire space. By Lemma 6.2(ii), we know that $C(b) \subseteq C$. On the other hand, let $x \in C$ and let $\epsilon > 0$. There exists $i \in \mathbf{N}$ such that $\|(b-a_i)+\tilde{J}_x\| < \epsilon/2$. Since $x \in C'(b-a_i)$, there is an open neighbourhood W of x in X_ϕ such that $\|(b-a_i)+\tilde{J}_y\| < \epsilon$ for all $y \in W$. Hence $\|(b-a_i)+H_x\| \leq \epsilon$ by [3, Lemma 4.2]. Since ϵ was arbitrary, $b \in H_x + A$, so $x \in C(b)$. Thus $C(b) = C$, which is a dense G_δ in X_ϕ.

Case (ii). Let $\{d_i\}_{i\geq 1}$ be a countable dense subset of $K(H)$ and write $X = \bigcup_{n=1}^{\infty} X_n$ where $\{X_n\}_{i\geq 1}$ is an increasing family of non-empty compact subsets of X. For each $(i,n) \in \mathbf{N}\times\mathbf{N}$, let $a_{(i,n)} : X \to K(H)$ be a continuous function taking the constant value d_i on X_n and vanishing at infinity. Then $a_{(i,n)} \in A$. For each $(i,n) \in \mathbf{N}\times\mathbf{N}$, let $C'(b-a_{(i,n)})$ be the set of points of continuity of the function $x \mapsto \|(b-a_{(i,n)})+\tilde{J}_x\|$ $(x \in X_\phi = X)$. Then $C'(b-a_{(i,n)})$ is a dense G_δ in X_ϕ by Lemma 6.2(i) and Proposition 6.3. Thus $C := \bigcap_{(i,n)\in\mathbf{N}\times\mathbf{N}} C'(b-a_{(i,n)})$ is a dense G_δ in X_ϕ since X_ϕ is a locally compact, Hausdorff space, and hence a Baire space. By Lemma 6.2(ii), we know that $C(b) \subseteq C$. On the other hand, let $x \in C$ and let $\epsilon > 0$. Since $b \in A^s$, $b(x) \in K(H)$, so there exists $(i,n) \in \mathbf{N}\times\mathbf{N}$ such that $\|(b-a_{(i,n)})+\tilde{J}_x\| < \epsilon/2$. Since $x \in C'(b-a_i)$, there is an open neighbourhood W of x in X_ϕ such that $\|(b-a_{(i,n)})+\tilde{J}_y\| < \epsilon$ for all $y \in W$. Hence $\|(b-a_{(i,n)})+H_x\| \leq \epsilon$ by [3, Lemma 4.2]. Since ϵ was arbitrary, $b \in H_x + A$, so $x \in C(b)$. Thus $C(b) = C$, which is a dense G_δ in X_ϕ. □

For $b \in A^s$ and $\epsilon > 0$, let

$$\mathrm{coz}_\epsilon(b+A) := \{x \in \beta X_\phi : \|b+(H_x+A)\| \geq \epsilon\} \subseteq X_\psi.$$

Then $b \in A$ if and only if $\mathrm{coz}_\epsilon(b+A) = \emptyset$ for all $\epsilon > 0$, and $b \in A^b$ if and only if $\mathrm{coz}_\epsilon(b+A) \subseteq \beta X_\phi \setminus X_\phi$ for all $\epsilon > 0$.

Theorem 6.5. *Let A be a continuous $C_0(X)$-algebra with base map ϕ. Suppose either that A is separable or that $A = C_0(X)\otimes K(H)$ where X is a σ-compact, locally compact Hausdorff space. Let $b \in A^s$ and let $\epsilon > 0$. Then*
(i) $\mathrm{coz}_\epsilon(b+A) = \{x \in \beta X_\phi : \|b+(H_x+A)\| \geq \epsilon\}$ *is a meagre compact set in βX_ϕ;*
(ii) $\mathrm{coz}(b+A)$ *is a meagre F_σ in βX_ϕ.*

Proof. We prove (i) and (ii) together. To see that $\mathrm{coz}_\epsilon(b+A)$ is compact, note that

$$\mathrm{coz}_\epsilon(b+A) = \psi(\{P \in \mathrm{Prim}(A^s/A) : \|(b+A)+P\| \geq \epsilon\}),$$

and the image of a compact set under a continuous map is compact. Then

$$\mathrm{coz}(b+A) = \bigcup_{n\geq 1} \mathrm{coz}_{1/n}(b+A),$$

which is an F_σ in βX_ϕ. Recall that since A is continuous, X_ϕ is a dense open subset of βX_ϕ. Hence by Lemma 6.4 the set $C(b)$ is a dense G_δ in βX_ϕ. But by Lemma 5.1,

$\mathrm{coz}(b + A) = (X_\phi \setminus C(b)) \cup \mathrm{coz}_\infty(b)$ and hence $\mathrm{coz}(b + A)$ is meagre. Thus its subset $\mathrm{coz}_\epsilon(b + A)$ is also meagre, as required. □

Note that $\mathrm{coz}(b+A)$ is not necessarily meagre in X_ψ, see Example 5.3(ii). However, $\mathrm{coz}(b+A)$ is meagre in X_ψ when X_ϕ has no isolated points, as we now show. We are grateful to Ramiro de la Vega for showing us the proof of (i) of the next result (on mathoverflow).

Corollary 6.6. *Let A be a continuous $C_0(X)$-algebra with base map ϕ. Suppose either that A is separable with A/J_x non-unital for all $x \in X_\phi$ or that $A = C_0(X) \otimes K(H)$ where X is a σ-compact, locally compact Hausdorff space. If X_ϕ has no isolated points then*
(i) *X_ψ is a Baire space;*
(ii) *for all $b \in A^s$, $\mathrm{coz}(b + A)$ is a meagre F_σ in X_ψ.*

Proof. (i) First note that in either case A is σ-unital, so that Theorem 5.2 applies. Let W be the set of all P-points in βX_ϕ and recall that $X_\psi = \beta X_\phi \setminus W$, Theorem 5.2. Note that if C is a non-empty G_δ subset of βX_ϕ then $C \nsubseteq W$. For if $p \in C \subseteq W$ then by definition of a P-point and by the local compactness of βX_ϕ, p has a compact neighbourhood in W. But a compact P-space is finite [10, 4K], and hence p is an isolated point of βX_ϕ, contradicting the hypotheses.

Now let $(U_n)_{n\geq 1}$ be a sequence of open dense subsets of X_ψ. Then each $U_n = V_n \cap X_\psi$ for some open set V_n in βX_ϕ. Each non-empty open subset of βX_ϕ is a G_δ and thus meets X_ψ by the previous paragraph. Hence X_ψ is dense in βX_ϕ so V_n is also dense in βX_ϕ. Thus $V := \bigcap_{n\geq 1} V_n$ is dense in βX_ϕ. Let U be any non-empty open subset of βX_ϕ. Then $U \cap V$ is a non-empty G_δ in βX_ϕ, and thus $U \cap V$ meets X_ψ by the previous paragraph. It follows that $\bigcap_{n\geq 1} U_n = V \cap X_\psi$ is dense in X_ψ, as required.

(ii) It is sufficient to show that for $b \in A^s$ and $n \geq 1$, the compact set $Y := \mathrm{coz}_{1/n}(b + A)$ has empty interior in X_ψ. With the notation of part (i), it is clear that Y does not meet W. Let U be an open subset of X_ψ such that $U \subseteq Y$ and let U' be an open subset of βX_ϕ such that $U' \cap X_\psi = U$. Then $U' \setminus U = U' \setminus Y$ which is open in βX_ϕ. But $U' \setminus U \subseteq W$, and must therefore be empty by the argument in the first paragraph of (i). Hence $U' = U$, so U is open in βX_ϕ, and U must therefore also be empty by Theorem 6.5(i). □

One point of interest in Theorem 6.5 and Corollary 6.6 is the sidelight that they cast on the work of Kasparov, Pedersen, Kucerovsky and others on separable and σ-unital C*-subalgebras of corona algebras, see [11], [16], [12]. For example, Pedersen showed if B is a σ-unital hereditary C*-subalgebra of the corona algebra of a σ-unital C*-algebra then $B = (B^\perp)^\perp$ [16, Theorem 15]. If A satisfies the hypotheses of Corollary 6.6 and if $((b_n+A))_{n\geq 1}$ is an approximate identity for a σ-unital C^*-subalgebra B of A^s/A then, by Corollary 6.6(ii), $\cup_{n\geq 1}\mathrm{coz}(b_n + A)$ is a meagre F_σ subset of X_ϕ. Thus B is a relatively small subalgebra of A^s/A. In particular, using Corollary 6.6(i), it follows that A^s/A itself is not σ-unital.

Indeed we can say more than this: if $\mathrm{Prim}(A^s/A)$ were σ-compact it would be Lindelof, and thus there would exist an element $b \in A^s$ with $b + A \notin P$ for all $P \in \mathrm{Prim}(A^s/A)$. Thus $\mathrm{coz}(b)$ would equal X_ψ, giving a contradiction between (i) and (ii) of Corollary 6.6. Hence $\mathrm{Prim}(A^s/A)$ is not σ-compact. Our final result gives some idea how far $\mathrm{Prim}(A^s/A)$ is from being σ-compact.

Recall that a topological space X is *weakly Lindelof* if any open cover has a countable subcover $\mathcal{V}$ such that $\bigcup\{V : V \in \mathcal{V}\}$ is dense in X. It is well known that if X satisfies the countable chain condition then X is weakly Lindelof. The following theorem is therefore also a partial extension of Theorem 6.1.

Theorem 6.7. *Let A be a continuous $C_0(X)$-algebra with base map ϕ. Suppose either that A is separable or that $A = C_0(X) \otimes K(H)$ where X is a σ-compact, locally compact Hausdorff space. If A/J_x is non-unital for all $x \in X_\phi$ and if X_ϕ has no isolated points then* $\mathrm{Prim}(A^s/A)$ *is not weakly Lindelof.*

Proof. For each $P \in \mathrm{Prim}(A^s/A)$ let $b_P \in A^s$ with $\|(b_P + A) + P\| = 1$. Let

$$V_P := \{Q \in \mathrm{Prim}(A^s/A) : \|(b_P + A) + Q\| > 1/2\}.$$

Then V_P is an open neighbourhood of P in $\mathrm{Prim}(A^s/A)$ so the set $\{V_P\}_{P\in\mathrm{Prim}(A^s/A)}$ is an open cover of $\mathrm{Prim}(A^s/A)$. Suppose for a contradiction that this cover has a countable subcover $\{V_i\}_{i\geq 1}$ such that $V := \bigcup_{i=1}^\infty V_i$ is dense in $\mathrm{Prim}(A^s/A)$. For each $i \geq 1$, let $b_i \in A^s$ such that

$$V_i = \{Q \in \mathrm{Prim}(A^s/A) : \|(b_i + A) + Q\| > 1/2\}.$$

Then $\psi(V_i) \subseteq \mathrm{coz}_{1/2}(b_i + A)$, which is a compact meagre set by Theorem 6.5(i).

Now let $\{Y_i\}_{i\geq 1}$ be a countable family of non-empty meagre closed subsets of βX_ϕ. Then $Y = \bigcup_{i=1}^\infty Y_i$ is a meagre F_σ subset of βX_ϕ. Set $R = \beta X_\phi \setminus X_\phi$. Then R is also a meagre closed subset of βX_ϕ so there exists $x \in \beta X_\phi \setminus (Y \cup R) \subseteq X_\phi$. For each $i \geq 1$, let $f_i \in C(\beta X_\phi)$ with $0 \leq f \leq 1$ such that $f_i(x) = 0$ and $f_i(Y_i \cup R) = 1$. Set $f' = \sum_{i=1}^\infty f_i/2^i$ and let Z' be the zero set of f' in βX_ϕ. Then $x \in Z'$ and $Z' \cap (Y \cup R) = \emptyset$. In particular, Z' is a compact subset of X_ϕ. We now consider two cases. If the boundary of Z' is empty then Z' is a non-empty clopen set of βX_ϕ. Since a compact P-space is finite [10, 4K], and X_ϕ has no isolated points, it follows that there is a non-P-point $y \in Z'$. Let $g : \beta X_\phi \to [0, 1]$ be a continuous function supported in Z' such that y lies in the boundary of the cozero set of g. Set $f = f' + g$ and let Z be the zero set of f. On the other hand, if the boundary of Z' is non-empty, then simply take $Z = Z'$. Either way, Z is a zero set in βX_ϕ with $Z \cap (Y \cup R) = \emptyset$ and with the boundary ∂Z of Z non-empty. It follows that Z is also a zero set in X_ϕ.

Now apply the argument of the previous paragraph in the case when $Y_i = \mathrm{coz}_{1/2}(b_i + A)$ ($i \geq 1$). With the resulting zero set Z, let $c^Z \in A^s$ be an element with the properties of Lemma 3.5. Let h be a continuous function on βX_ϕ with $0 \leq h \leq 1$ such that $h(R) \subseteq \{0\}$ and $h(Z) \subseteq \{1\}$. By the Dauns-Hofmann Theorem, the continuous function $h \circ \overline{\phi}$ on $\mathrm{Prim}(M(A))$ induces an element k in the centre of $M(A)$ such that $k + H_x = h(x)$ for all $x \in \beta X_\phi$. Set $d = kc^Z \in A^s$. Then because h is 1 on Z, $C(d) = C(c^Z) = X_\phi \setminus \partial Z$ (using Lemma 3.6). In particular, since ∂Z is non-empty, $d \notin A$. Then, by Lemma 5.1 and the fact that $h(R) \subseteq 0$, $\mathrm{coz}(d + A) = X_\phi \setminus C(d) = \partial Z$. Since $\psi(V) \subseteq Y$, we have that

$$\{Q \in \mathrm{Prim}(A^s/A) : (d + A) \notin Q\}$$

is a non-empty open subset of $\mathrm{Prim}(A^s/A)$ disjoint from V. This gives the required contradiction. $\square$

In particular, in the context of Theorem 6.7 it follows that $\mathrm{Prim}(A^s/A)$ is not Lindelof. Since a closed subset of a Lindelof space is Lindelof, this implies that $\mathrm{Prim}(A^s)$ is not Lindelof and hence not σ-compact (recall that a locally compact space is Lindelof if and only if it is σ-compact). On the other hand, for any σ-unital C*-algebra A, $\mathrm{Prim}(A)$ is a dense σ-compact subset of $\mathrm{Prim}(A^s)$ which easily implies that $\mathrm{Prim}(A^s)$ is weakly Lindelof.

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